Exercise 84

Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that has heard the rumor at time t and a and k are positive constants. [In Section 9.4 we will see that this is a reasonable equation for $p(t)$.]

(a) Find $\lim_{t\to\infty} p(t)$.

(b) Find the rate of spread of the rumor.

(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

Solution

Take the limit of $p(t)$ as $t \to \infty$.

$$\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a(0)} = 1$$

What this means is that everyone will have heard the rumor after a really long time.
The rate of spread of the rumor is given by the derivative of $p(t)$.

$$\begin{aligned}
p'(t) &= \frac{dp}{dt} \\
&= \frac{d}{dt}\left(\frac{1}{1 + ae^{-kt}}\right) \\
&= \frac{\left[\frac{d}{dt}(1)\right](1 + ae^{-kt}) - \left[\frac{d}{dt}(1 + ae^{-kt})\right](1)}{(1 + ae^{-kt})^2} \\
&= \frac{(0)(1 + ae^{-kt}) - \left[a\frac{d}{dt}(e^{-kt})\right]}{(1 + ae^{-kt})^2} \\
&= \frac{-\left[a(e^{-kt}) \cdot \frac{d}{dt}(-kt)\right]}{(1 + ae^{-kt})^2} \\
&= \frac{-\left[a(e^{-kt}) \cdot (-k)\right]}{(1 + ae^{-kt})^2} \\
&= \frac{kae^{-kt}}{(1 + ae^{-kt})^2}
\end{aligned}$$

To find how long it will take for 80% of the population to hear the rumor, set $p(t) = 0.8$ and solve for t.

$$0.8 = \frac{1}{1 + ae^{-kt}}$$

$$0.8(1 + ae^{-kt}) = 1$$

Divide both sides by 0.8.

$$1 + ae^{-kt} = \frac{1}{0.8}$$

Subtract both sides by 1.

$$ae^{-kt} = \frac{1}{0.8} - 1 = \frac{1}{4}$$

Divide both sides by a.

$$e^{-kt} = \frac{1}{4a}$$

Take the natural logarithm of both sides.

$$\ln e^{-kt} = \ln \frac{1}{4a}$$

$$-kt \ln e = \ln \frac{1}{4a}$$

Solve for t.

$$t = -\frac{1}{k} \ln \frac{1}{4a} = \frac{1}{k} \ln 4a$$

Assuming $a = 10$ and $k = 0.5$,

$$t = \frac{1}{0.5} \ln 40 \approx 7.38 \text{ hours.}$$

The plot of $p(t)$ versus t below confirms this result.